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SOME NON-UNIFORMLY HOMEOMORPHIC SPACES

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ABSTRACT

In this paper we prove that for $0 < p, q \leq 1$ the real F-spaces $L_q[0,1]$ and ℓ_p are not uniformly homeomorphic. The particular case $p = q = 1$ is due to Enflo and our work is motivated by his.

1. Introduction

It was proved by Enflo (unpublished) that $L_1[0,1]$ and ℓ_1 are not uniformly homeomorphic. An account of Enflo's proof can be found in Benyamini's expository paper [1]. As with many results in the Uniform Theory, Enflo's proof used the following basic lemma:

A uniformly continuous map f from a metrically convex metric space (M_1, ρ_1) into a metric space (M_2, ρ_2) satisfies a Lipschitz condition of order 1 for large distances (i.e., given $\delta > 0$, there is a constant $F(\delta)$ so that $\rho_2(f(x), f(y)) \leq F(\delta)\rho_1(x,y)$ whenever $\rho_1(x,y) \geq \delta$).

Note that a metric space (M, ρ) is said to be **metrically convex** if given points $x \neq y$ in M we can always find a point z in M such that $\rho(x,z) = \frac{1}{2}\rho(x,y) = \rho(z,y)$. Such a point z is called a **metric midpoint** between x and y in (M, ρ).

In modifying Enflo's original argument we encounter an obstacle that relates to our intended use of the stated basic lemma. The problem is that in the

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setting $0 < p < 1$ the real F-space ℓ_p, with its usual metric d (ie., for $x = (x_j)$ and $y = (y_j)$ in ℓ_p, $d(x,y) := \sum_{j=1}^{\infty} |x_j - y_j|^p$), is not metrically convex. So the basic lemma doesn't immediately apply. Indeed, uniformly continuous maps $f : (\ell_p, d) \to (M, \rho)$ satisfy, in general, only Lipschitz conditions of order $1/p$ for large distances, a statement too weak for our purposes. We deal with this shortfall in the next section by introducing a uniformly equivalent remetrisation of ℓ_p that lends itself to an argument along Enflo's original lines.

It is worth noting that the unit balls of the spaces appearing in our abstract are uniformly homeomorphic. This follows easily from estimates in Mazur's 1929 Studia paper [2].

Some of the details in the proof of Theorem 4 are analogous to the original arguments of Enflo. They are included not only for completeness but also because [1] is not a regular publication and may be hard to obtain.

Throughout $0 < p$, $q \leq 1$ and d denotes the usual metric for both $L_q[0,1]$ and ℓ_p.

2. Lemmas and Main Result

We begin by remetrising ℓ_p according to the formula

$$d_1(x,y) := \sum_{j=1}^{\infty} \varphi(|x_j - y_j|); \quad x = (x_j),\ y = (y_j) \text{ in } \ell p,$$

where

$$\varphi(t) := \begin{cases} t, & \text{if} \quad t \geq 1, \\ t^p, & \text{if} \quad 0 < t \leq 1. \end{cases}$$

This metric gives the same uniform structure as the usual metric d on ℓ_p because $d_1(x,y) = d(x,y)$ whenever $d(x,y) \leq 1$.

It is useful to note that that d_1 can also be realised abstractly in a way that sometimes aides our intuition. A **1-chain** between points x and y in ℓ_p is a finite set of points

$x = x_0, x_1, \ldots, x_n = y$ in ℓ_p such that $d(x_j, x_{j+1}) \leq 1$ for $j = 0, 1, \ldots, n-1$. The d-**length** of this 1-chain is the quantity

$$\ell(x_0, x_1, \ldots, x_n) := \sum_{j=0}^{n-1} d(x_j, x_{j+1}).$$

An easy (but tedious) argument establishes that $d_1(x,y)$ equals the infimum of the d-lengths of the 1-chains between x and y.

An ad hoc explanation (of the virtues) of the d_1 metric is that it convexifies ℓ_p for large distances (without altering the uniform structure). Precisely what is meant will become clear in the proof of our first lemma.

LEMMA 1: *Suppose that (M,ρ) is a metric space and that $f:\ell_p\to(M,\rho)$ is a uniformly continuous map. Then, relative to the d_1 metric, f satisfies a Lipschitz condition of order 1 for large distances (>2). In other words: given $\delta>2$, there is a constant $F(\delta)$ such that*

$$\rho(f(x),f(y))\le F(\delta)d_1(x,y) \text{ whenever } d_1(x,y)\ge\delta.$$

Proof: Let $\delta>2$ be given. Then there is a constant M so that $\rho(f(a),f(b))\le M$ whenever $d_1(a,b)<\delta$. Indeed, by the uniform continuity of f there is a δ_1 so that $d_1(x,y)<\delta_1$ implies that $\rho(f(x),f(y))<1$. Now divide the interval $[a,b]$ into M equal parts $a=a_0<\cdots<a_M=b$, where $d(a_i,a_{i+1})<\delta_1$, and $M\le 2(\delta/\delta_1)^{1/p}$. Then $\rho(f(a),f(b))\le\sum_{i=0}^{M-1}\rho(f(a_i),f(a_{i+1}))\le M$.

Let x and y be given points in ℓ_p with $d_1(x,y)\ge\delta$. Let m be the largest integer such that $m\delta\le d_1(x,y)$. We may choose a 1-chain $x=x_0,x_1,\dots,x_n=y$ with $\ell(x_0,x_1,\dots,x_n)<(m+1)\delta$.

Since $1<\frac{\delta}{2}$ it follows that we may choose integers $0=k_0<k_1<\ldots<k_l=n$ such that

$$\frac{\delta}{2}\le\ell(x_{k_j},x_{k_j+1},\dots,x_{k_{j+1}})\le\delta \text{ for } j=0,1,\dots,l-2,$$

and

$$0<\ell(x_{k_{l-1}},x_{k_{l-1}+1},\dots,x_{k_l})\le\delta.$$

Notice that $d_1(x_{k_j},x_{k_{j+1}})\le\delta$ for $j=0,1,\dots,l-1$, and also that

$$(l-1)\frac{\delta}{2}\le\ell(x_0,x_1,\dots,x_n)<(m+1)\delta$$

from whence it follows that $l<5m$ (because $m\ge1$). Thus

$$\begin{aligned}\rho(f(x),f(y)) &\le\sum_{j=0}^{l-1}\rho(f(x_{k_j}),f(x_{k_{j+1}}))\\ &\le lM\\ &<5mM.\end{aligned}$$

Therfore

$$\rho(f(x), f(y)) \leq \frac{5M}{\delta} d_1(x,y)$$

which completes the proof. ∎

Given points a and b in ℓ_p and $0 < \epsilon < \frac{1}{4}$ we define the set

$$V_\epsilon(a,b) := \{v \in \ell_p | \frac{1-4\epsilon}{2} d_1(a,b) \leq d_1(a,v), d_1(b,v) \leq \frac{1+4\epsilon}{2} d_1(a,b)\}$$

of d_1-**almost metric midpoints** between a and b. Using the closed formula for d_1 it is routine to check

LEMMA 2: *Given points a and b in ℓ_p, $V_\epsilon(a,b)$ is contained in the set $[a,b] + B_{4\epsilon d_1(a,b)}$ where $[a,b]$ is the compact lattice interval of those points in ℓ_p whose coordinates lie between the corresponding coordinates of a and b.*

LEMMA 3: *Given points x and y in $L_q[0,1]$ there exists a sequence (x_n) in $L_q[0,1]$ of metric midpoints between x and y with the property that $d(x_j, x_k) = \frac{1}{2}d(x,y)$ whenever $j \neq k$.*

Proof: We only indicate the essential idea of the construction (leaving the straightforward inductive details out of this paper).

By a translation we may assume that $x \equiv 0$. It is a standard fact that the function

$$F(s) := \int_{[0,s]} |y(t)|^p dt$$

is continuous on $[0,1]$ and, obviously, F is monotone with $F(0) = 0$ and $F(1) = d(y,0)$.

Choosing $\alpha_1 \in (0,1)$ such that $F(\alpha_1) = \frac{1}{2}d(y,0)$ we set

$$x_1 \equiv \begin{cases} y & \text{on } [0,\alpha_1], \\ 0 & \text{otherwise.} \end{cases}$$

Choosing $\alpha_2 \in (0,1)$ such that $F(\alpha_2) = \frac{1}{2^2}d(y,0)$ and $\alpha_3 \in (\alpha_1, 1)$ such that $F(\alpha_3) - F(\alpha_1) = \frac{1}{2^2}d(y,0)$ we set

$$x_2 \equiv \begin{cases} y & \text{on } [0,\alpha_2] \cup [\alpha_1,\alpha_3], \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that $d(x_1, x_2) = \frac{1}{2}d(y,0)$. The construction now goes through in the obvious fashion. Notice that if $y \equiv 1$ then $x_n = (1 + r_n)/2$ where $r_1, r_2, \ldots$ are the non-trivial Rademacher functions on $[0,1]$. ∎

THEOREM 4: *The real F-spaces $L_q[0,1]$ and ℓ_p are not uniformly homeomorphic.*

Proof: Assume to the contary that $f : L_q \to \ell_p$ is a uniform homeomorphism.

f^{-1} is uniformly continuous so there is a constant $\gamma > 0$ such that $d(x,y) \leq \gamma$ whenever $d_1(f(x), f(y)) \leq 2$. Now if $d(x,y) > \gamma$ then it must be that $d_1(f(x), f(y)) > 2$ and so by Lemma 1 (applied to f^{-1}) there must be an $L > 0$ such that $d(x,y) \leq L d_1(f(x), f(y))$. In other words, we have constants $\gamma > 0$ and $L > 0$ such that

$$d(x,y) \leq \max\{\gamma, L d_1(f(x), f(y))\} \quad \text{for all points } x \text{ and } y \text{ in } L_q.$$

As L_q is metrically convex we know that f satisfies a Lipschitz condition of order 1 for large distances. Hence, given $\delta > 0$, there is a smallest constant $K(\delta)$ such that $d_1(f(x), f(y)) \leq K(\delta) d(x,y)$ whenever $d(x,y) \geq \delta$.

$K(\delta)$ decreases as δ increases so we may set $K := \lim_{\delta\to\infty} K(\delta)$.

Given $0 < \epsilon < \frac{1}{4}$ we can choose $\delta > \gamma$ so that $K(\delta) \leq (1+\epsilon)K$.

The minimality of $K(2\delta)$ allows us to fix points x and y in L_q with $d(x,y) \geq 2\delta$ and

$$d_1(f(x), f(y)) \geq (1-\epsilon)K(2\delta)d(x,y) \geq (1-\epsilon)Kd(x,y). \tag{1}$$

We set $U := \{u \in L_q |\ d(x,u) = \frac{1}{2}d(x,y) = d(y,u)\}$, the set of metric midpoints between x and y in L_q, and we observe that $f(U) \subseteq V_\epsilon(f(x), f(y))$ (notation as before). Indeed, if $u \in U$, then

$$\begin{aligned} d_1(f(x), f(u)) &\leq K(\delta)d(x,u) \\ &= \tfrac{1}{2}K(\delta)d(x,y) \\ &\leq \tfrac{1}{2}(1+\epsilon)Kd(x,y) \\ &\leq \tfrac{1+\epsilon}{2(1-\epsilon)} d_1(f(x), f(y)) \quad \text{by (1).} \\ &\leq \tfrac{1+4\epsilon}{2} d_1(f(x), f(y)), \end{aligned}$$

this last inequality being a result of having $0 < \epsilon < 1/4$. The remaining details of the containment $f(U) \subseteq V_\epsilon(f(x), f(y))$ are similar and are omitted from this paper.

By Lemma 3, U contains an infinite sequence (x_n) such that $d(x_j, x_k) = \frac{1}{2}d(x,y)$ whenever $j \neq k$ and, by Lemma 2 and the above containment, $f(U) \subseteq [f(x), f(y)] + B_{4\epsilon d_1(f(x),f(y))}$. The compactness of the lattice interval tells us that there are distinct indices j and k such that $d_1(f(x_j), f(x_k)) \leq 10\epsilon d_1(f(x), f(y))$, and as $d(x_j, x_k) = \frac{1}{2}d(x,y) \geq \delta > \gamma$ we obtain:

$$\begin{aligned} d(x,y) &= 2d(x_j, x_k) \\ &\leq 2Ld_1(f(x_j), f(x_k)) \\ &\leq 20\epsilon L d_1(f(x), f(y)) \\ &\leq 20\epsilon L K(\delta) d(x,y) \\ &\leq 20\epsilon L(1+\epsilon) K d(x,y) \end{aligned}$$

So $1 \leq 20\epsilon L(1+\epsilon)K$ and a contradiction is found letting $\epsilon \to 0$. ∎

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